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Stochastically-driven coherence in a sine-Gordon chain

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We perform numerical simulations of the dynamical behavior of a sine-Gordon chain in a heat bath. The interaction with the heat bath is simulated by the Langevin formalism. The noise term is uncorrelated in both space and time. We use the Karhunen-Loeve decomposition to study the effective number of degrees of freedom as a function of temperature (i.e., of the noise dispersion). At low temperatures we find a spatially disordered regime, characterized by a high number of degrees of freedom. At a temperature of the order of the soliton rest mass we find a relatively sharp crossover to an ordered regime, characterized by a low number of degrees of freedom. The spatial structure of the modes suggests that the transition is associated to the appearance of thermally activated solitons. We also present an alternative estimate of the effective number of degrees of freedom.

The sine-Gordon-like equations model a wide variety of physical situations. This occurs frequently in solid-state physics because the sine-Gordon equation is the simplest wave-equation for a periodic medium. Systems modeled by the driven damped sine-Gordon equation display many of the phenomena commonly associated to nonlinear behavior, such as coexistence of solitons and chaos [1], and turbulence [2]. Even richer behavior can also be exhibited by sine-Gordon-like systems, as noise induced pattern formation.

In this paper we study a sine-Gordon chain in a heat bath in a range of temperatures. We find a sharp crossover as a function of temperature with two different definitions of the effective number of degrees of freedom. The results are analyzed by means of the Karhunen-Loeve [3] decomposition to examine the variation of the effective number of degrees of freedom. The Karhunen-Loeve analysis provides an estimate of the number of degrees of freedom as the number of modes containing an arbitrary fraction of the correlation [4] and has been used by Sirovich [5] and Knight [6] as a procedure for the determination of coherent structures. As this estimate may become unreliable when the weight distribution is highly inhomogeneous, we also discuss an alternative estimate of the number of degrees of freedom that contains no arbitrary cutoff and partially accounts for inhomogenieties in the weight distribution.

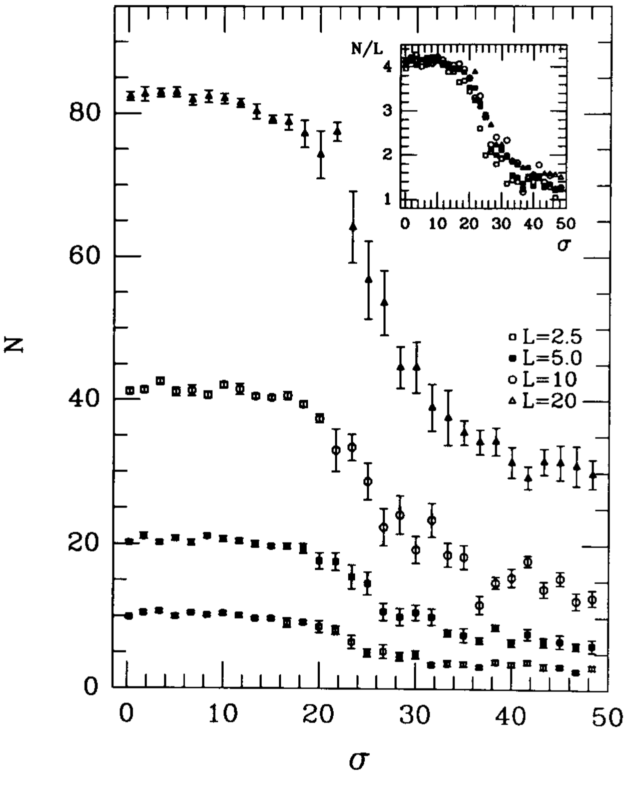

Figure 1. *Degrees of freedom vs. thermal noise.* Here N is the number of modes that contains 99.99% of the weight, $W = \sum \lambda_n$, and σ is the variance of the noise. Averages are performed over ten realizations. The inset shows the curves re-scaled with the length.

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The sine-Gordon equation $\phi_{xx} - \phi_{tt} - \sin\phi = 0$, models a chain of coupled pendula. We couple this chain of pendula to a heat bath following the Langevin approach:

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t - R(x,t) \qquad (1)$$

We measure all quantities in their natural units, so our equations are dimensionless. The first term on the right-hand side of Eq. (1) is the loss term representing the energy flow to the reservoir, while the second term is the noise associated with α, giving the disordered thermal-energy input to the system. The noise term is "white" both in space and time. The effect of the random force $R(x,t)$ is "to heat" the pendula: a soliton-like excitation can appear when a given pendulum escapes from its potential well. We use flat initial conditions ($\phi(x,0) = \phi_t(x,0) = 0$) and open boundary conditions ($\phi_x(0,t) = \phi_x(L,t) = 0$). The parameters of our simulations are $\alpha = 0.252$, $\Delta x = 0.039$ and $\Delta t = 0.035$.

The Karhunen-Loeve decomposition allows us to describe the dynamics in terms of an adequate basis of orthonormal functions or modes. The field $\omega(x,t)$ to be decomposed represents the fluctuations of $\phi(x,t)$ with respect to the time-averaged spatial pattern $\phi_{ave}(x)$. We find a basis of orthonormal functions $\Psi_n(x)$ by solving an integral equation whose kernel is the two points correlation function $K(x,x') = \langle\omega(x,t)\omega(x',t')\rangle$ (here $\langle\ldots\rangle$ means time average). The functions $\Psi_n(x)$ are the eigenfunctions of the integral equation,

$$\int_0^L K(x,x')\Psi_n(x')dx' = \lambda_n \Psi_n(x) \qquad (2)$$

The eigenvalues λ_n can be regarded as the weight of the "mode" n; we can estimate the number of modes effectively contributing to the dynamics as an arbitrary percentage of the total weight $W = \sum \lambda_n$.

In Figure 1 we present for sine-Gordon chains of different lengths the sharp crossover related with the thermal activation of solitons. It can be seen that the error bars increase as the system experiences the dynamical transition to the solitonic regime.

Alternatively we can define a "configuration entropy":

$$S \sim -\sum_n P_n \ln P_n ; P_n = \frac{\lambda_n}{\sum_n \lambda_n} \qquad (3)$$

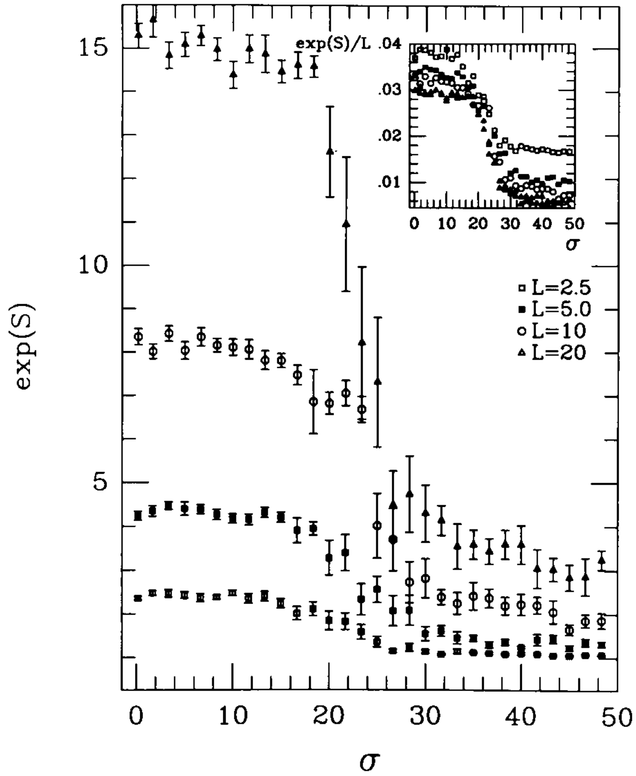

Figure 2. "Configuration entropy" based treatment.

In Figure 2 we present $exp(S)$ as an estimate of the effective number of degrees of freedom plotted as varies the variance of the noise; this estimate removes the arbitrary selection of percentage of weight contained by the modes. The inset also shows an extrapolation to infinite length (denoted with filled triangles).

The noise induced pattern phenomenon in our system has been also qualitatively characterized.

REFERENCES

1. A. R. Bishop and P. S. Lomdahl, Physica D 18 (1986) 54
2. L. E. Guerrero and M. Octavio, Phys. Rev. A 42 (1990) 4630
3. K. Karhunen, Ann. Acad. Sci. Fennicae, Ser A1, Math. Phys. 37 (1946); M. M. Loeve, Probability Theory , Van Nostrand, Princeton, 1955.
4. S. Ciliberto and B. Nicolaenko, Europhys. Lett. 14 (1991) 303.
5. L. Sirovich, Physica D 37 (1989) 126.
6. B. Knight and L. Sirovich, Phys. Rev. Lett. 65 (1990) 1365.