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Physica A 257 (1998) 390–394

PHYSICA A

Long-range interacting solitons: pattern formation and nonextensive thermostatistics

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Abstract

The nonlinear Klein–Gordon equation with a different potential that satisfies the degeneracy properties discussed in this paper possesses solitonic solutions that interact with long-range forces. We generalize the Ginzburg–Landau equation in such a way that the topological defects supported by this equation present long-range interaction both in $D=1$ and $D>1$. Finally, we construct a system of two equations with two complex order parameters for which the interaction forces between the topological defects decay so slowly that the system enters the nonextensivity regime.

PACS: 03.40.Kf; 05.40.+j; 61.43.Hv; 64.60.Fr
Keywords: Solitons; Long-range interactions; Pattern formation

1. Introduction

There is a great interest in the formulation of models in which the solitons can interact with long-range forces [1–6]. The soliton solutions of the well-known models (e.g., sine-Gordon and ϕ^4 equations) interact with short-range forces [1]. There is experimental evidence [4,5] that most real transfer mechanisms have long-range character. We are interested in models where there is spontaneous formation of particle-like objects that possess long-range interactions. In such systems we can study pattern formation and other complex phenomena. Due to the fact that systems with long-range interactions can exhibit nonextensive behavior [7–9], the models we are investigating are relevant for the recently proposed thermostatistical theories [7].

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PII: S0378-4371(98)00165-4

Some authors have considered long-range effects [4] including ad hoc nonlocal terms in the equations. Spin systems have been also studied considering the coupling constant J_{ij} between the lattice spins to be proportional to $r_{ij}^{-\alpha}$ [9]. González and Estrada–Sarlabous demonstrated [1] that pure Klein–Gordon equations without coordinate-dependent terms, can support solitons with long-range interactions.

In this paper we investigate a system that is an extension of the Klein–Gordon equation and possesses soliton solutions with long-range interaction. We also introduce a generalized version of the Ginzburg–Landau equation which supports topological defects whose interaction force decays very slowly. It is possible to create a gas of such topological defects with an interaction force that decays so slowly that we enter the nonextensivity regime. We apply these results to nonequilibrium systems, pattern formation and growth models.

2. Modified Klein–Gordon equation

In this section we will study the Klein–Gordon equation

$$\phi_{tt} - \phi_{xx} - G(\phi) = 0\,. \tag{1}$$

Here $G(\phi) = -\partial U(\phi)/\partial\phi$. In Refs. [1,10] the authors investigated systems of type (1) where $U(\phi)$ possesses at least two minima (at points ϕ_1 and ϕ_3) and a maximum at point ϕ_2 ($\phi_1 < \phi_2 < \phi_3$). In particular, it was shown that if the potential $U(\phi)$ behaves in the neighborhood of a minima as $U(\phi) \sim (\phi - \phi_i)^{2n}$, then the solitons supported by the system interact with a force $F(d)$ that decays exponentially with the distance for $n = 1$. On the other hand, for $n > 1$ the solitons interact with a force that decays following the law $F \sim d^{2n/1-n}$.

2.1. Pattern formation

The most investigated growth model is the KPZ equation [11]. Nevertheless, other models have been proposed [12] including the sine-Gordon model [13,14].

In this section we present an alternative model which is given by the equation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - G(\phi) = \eta(x,t)\,, \tag{2}$$

where $\eta(x,t)$ is spatiotemporal white noise with the properties $\langle\eta(x,t)\rangle = 0$, $\langle\eta(x,t)\eta(x',t')\rangle = 2D\delta(t-t')\delta(x-x')$. Here the potential is $U(\phi) = 2\sin^{2n}(\phi/2)$. It can be shown that this potential has the property $U(\phi) \sim (\phi - \phi_i)^{2n}$. This equation with $n > 1$ can be used as a growth model for periodic media with marginal stability.

The growth model described by Eq. (2) presents noise-induced pattern formation that can be studied through the calculation of the roughness exponent. The sine-Gordon equation ($n = 1$) does not eliminate disorder at great scales [14]. Meanwhile, the systems with $n \geqslant 1$ display self-affine behavior at all scales [6]. The long-range interaction between the solitons is the most relevant feature of this system. This also can explain the wavelet analysis performed in Ref. [6]. There for $n \gg 1$ the authors found the existence of structures at all scales.

Summarizing, in this system there is pattern formation produced by the spontaneous generation of topological objects with long-range interactions which can create complex structures exhibiting fractal behavior.

3. A generalized Ginzburg–Landau equation

In this section we introduce a generalized version of the Ginzburg–Landau equation:

$$\frac{\partial u}{\partial t} = \nabla^2 u + u(1 - |u|^2)^{2n-1} . \tag{3}$$

Note that for $n=1$ we recover the well-known Ginzburg–Landau (G–L) equation [15,16]. Eq. (3) preserves all the qualitative properties of the G–L equation even for $n>1$. There exist an unstable state at $u=0$ and a degenerate stable state at $|u|=1$. For all integer n, Eq. (3) possesses topological solitons. However, for $D=1$ $(n=1)$ G–L solitons interact with a force that decays exponentially. On the other hand, the soliton solutions of Eq. (3) $(n>1)$ interact with long-range forces as the modified sine-Gordon equation.

Nevertheless, here we will be interested in vortice-like topological defects in $D=2$. A vortice-like topological defect with topological charge κ can be expressed in polar coordinates (r, φ) by the following equation $u=\rho(r)e^{i\kappa\varphi}$, where $\rho(r)$ is a solution of the equation

$$\frac{\partial^2 \rho}{\partial r^2} + \frac{1}{r}\frac{\partial \rho}{\partial r} - \frac{\kappa^2 \rho}{r^2} + \rho(1-\rho^2)^{2n-1} = 0 . \tag{4}$$

The analysis of the asymptotic behavior of the vortice solution for $n=1$ of Eq. (4) yields that in the limit $r \to \infty$, $(\rho(r)-1) \sim r^{-2}$. If $n>1$, then $(\rho(r)-1) \sim r^{1/1-n}$. Note that the transition from $n=1$ to $n>1$ is not continuous because for the long-range potentials that we are considering [1,6,17] n is usually taken as an integer number. In principle, n can be generalized to be a real number. In this case, we define $n=1+\varepsilon$. Hence for $\varepsilon<\frac{1}{2}$ we have $(\rho(r)-1) \sim r^{-2}$. On the other hand, for $\varepsilon>\frac{1}{2}$, $(\rho(r)-1) \sim r^{-1/\varepsilon}$. As we can see the solutions for $\varepsilon<\frac{1}{2}$ and $\varepsilon>\frac{1}{2}$ match if ε is real.

In the case of the generalized G–L, Eq. (3), the force that acts on a vortex situated at the point r due to the existence of another vortex at the coordinates origin satisfies the relation $F \sim (\rho(r)-1)$. Thus the vortices produced by Eq. (3) with $n=1$ have Coulomb interactions. Meanwhile, for $n>1$ the interaction decays much more slowly.

4. Nonextensivity

In principle, it is possible to construct a system described by equations of the type

$$\frac{\partial^2 \phi_1}{\partial t^2} + \gamma \frac{\partial \phi_1}{\partial t} - \nabla^2 \phi_1 = -\frac{\partial V(|\phi_1|, |\phi_2|)}{\partial \phi_1}\,, \tag{5}$$

$$\frac{\partial^2 \phi_2}{\partial t^2} + \gamma \frac{\partial \phi_2}{\partial t} - \nabla^2 \phi_2 = -\frac{\partial V(|\phi_1|, |\phi_2|)}{\partial \phi_2}\,, \tag{6}$$

where the potential $V(|\phi_1|, |\phi_2|)$ holds the necessary conditions in order to produce long-range interactions. When we are in the presence of systems of equations like Eqs. (5) and (6) with two order parameters we can have the situation where the sustained topological defects repel each other at very small distances and they attract each other at great distances [17]. We can have an effective interaction potential like the following:

$$V_{eff}(r) = \varepsilon \left[\left(\frac{\sigma}{1+r^2} \right)^{\rho/2} - \left(\frac{\sigma}{1+r^2} \right)^{\alpha/2} \right], \tag{7}$$

where $\alpha < \rho$. This is a situation equivalent to that discussed in Ref. [8]. Thus, when we have N particles in the system, the energy will grow with N following the laws:

$$E \sim \begin{cases} N & \text{if } \frac{\alpha}{D} > 1\,, \\ N \ln N & \text{if } \frac{\alpha}{D} = 1\,, \\ N^{2-\alpha/D} & \text{if } \frac{\alpha}{D} < 1\,. \end{cases} \tag{8}$$

In our case $\alpha = (2-n)/(n-1)$. When $n > 1$ (for n integer and $D = 2$) we are in the nonextensive regime. In the general case, for $n = 1 + \varepsilon$ (real) we obtain the nonextensivity condition $(1-\varepsilon)/\varepsilon < D$.

Acknowledgements

This work has been partially supported by Consejo Nacional de Investigaciones Científicas y Tecnológicas (CONICIT) under Project S1-2708.

References

[1] J.A. González, J. Estrada-Sarlabous, Phys. Lett. A 140 (1989) 189.
[2] P.S. Rica, E. Tirapegui, Phys. Rev. Lett. 64 (1990) 878.
[3] I.S. Aranson, L. Kramer, A. Weber, Phys. Rev. Lett. 67 (1991) 404.
[4] Yu.B. Gaididei, S.F. Mingaleev, P.L. Christiansen, K.O. Rasmussen, Phys. Lett. A 222 (1996) 152.
[5] A. Scott, Phys. Rep. 217 (1992) 1.
[6] L.E. Guerrero, E. López-Atencio, J.A. González, Phys. Rev. E 55 (1997) 7691.
[7] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[8] P.J. Jund, S.G. Kim, C. Tsallis, Phys. Rev. B 52 (1995) 50.

[9] S.A. Cannas, F.A. Tamarit, Phys. Rev. B 54 (1996) R12661.
[10] J.A. González, J.A. Hołyst, Phys. Rev. B 35 (1987) 3643.
[11] M. Kardar, G. Parisi, Y.-C. Zhang, Phys. Rev. Lett. 56 (1986) 889.
[12] F. Family, T. Vicsek (Eds.), Dynamics of Fractal Surfaces, World Scientific, Singapore, 1991; A. Barabási, H.E. Stanley, Fractal Concepts in Surface Growth, Cambridge University Press, Cambridge, 1995.
[13] J. Krug, H. Spohn, Europhys. Lett. 8 (1989) 219.
[14] R. Rangel, L.E. Guerrero, Fractals 3 (1995) 533.
[15] M.C. Cross, P.C. Hohenberg, Rev. Mod. Phys. 65 (1993) 851 and references therein.
[16] L.M. Pismen, J.D. Rodríguez, Phys. Rev. A 42 (1990) 2471.
[17] J.A. González, M. Martín-Landrove, J.R. Carbó, M. Chacón, ICTP Report No IC/94/301, Trieste, 1994.