Crossover Length from Invasion Percolation to Diffusion-Limited Aggregation in Porous Media

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We model fluid-fluid displacement in $d=2$ by a diffusion-limited-aggregation (DLA) algorithm which takes random capillary forces into account. Interpore surface tension is neglected. The invading fluid is nonviscous. We find a crossover length L_c. On length scales much smaller (larger) than L_c, invasion percolation (DLA) patterns are obtained. We argue by scaling, and check by simulations, that $L_c \sim (\Delta\tilde{p}/\mathrm{Ca})^{2/(2+D_s)}$; $\Delta\tilde{p}$ stands for a measure of spatial variations of the capillary pressure, Ca is the capillary number, and D_s is the interface fractal dimension on small length scales (we find $D_s \approx 1.3$).

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Diffusion-limited aggregation [1] (DLA) and invasion percolation [2] (IP) are two leading models of fractal growth [3]. DLA models, stochastically, the evolution of surfaces which advance with speed v fulfilling $v \propto \nabla p$, where p is some function satisfying Laplace's equation ($\nabla^2 p = 0$). Great interest has developed in such evolutions, known as Laplacian growth [4], because it is a fairly common phenomenon which shows up in various seemingly different circumstances [5]. The Laplacian growth which takes place when a nonviscous liquid is injected under high pressure into a porous medium already filled with a viscous fluid is an important practical problem in secondary oil recovery [6]. The standard DLA model is appropriate for (a) vanishing viscosity of the invading fluid (since Laplace's equation is only simulated on the space of the defending fluid), and (b) constant pressure on the viscous side of the interface (since no diffusive particles are generated on the interface in "standard" DLA) [7]. Neither capillary forces nor surface-tension effects are therefore accounted for. It is applicable, consequently, to very high injection rate (of zero viscosity liquids) experiments, when *viscous forces* in the displaced fluid are dominant. We refer to this regime as the DLA, or *viscous fingering,* or *open branch* (because *no* viscous fluid encirclement occurs then), regime. On the other hand, capillary forces dominate when the nonviscous fluid is injected very slowly into a porous medium previously filled with a viscous liquid. There is then widespread trapping [8] (i.e., encirclement) of the defending fluid; we shall refer to this regime as the *invasion percolation* regime. The IP model [2] has been designed for such a regime: The interface is advanced wherever the capillary pressure $\tilde{p}(r)$ is most favorable for it, neglecting viscous forces completely.

It is not known how the transition between the IP and the open branching regimes takes place as the injection rate is varied. Lenormand [9] has reported continuous evolution from IP to DLA in experiments on interconnected channels. Blunt and King [10] and others [11] have solved the corresponding Laplace equations numerically. However, a crossover length L_c, as a function of injection rate, between IP and DLA regimes has not been found [12].

It is the purpose of this Letter (a) to argue and to show through simulations that an L_c exists, such that DLA patterns obtain on length scales much larger than L_c, but IP-like growth takes place on length scales much smaller than L_c; and (b) to find how L_c behaves, as the injection rate varies, by scaling and by simulations. This Letter is arranged as follows. We first outline the model of porous media and of the appropriate DLA algorithm we use. The model (1) takes into account capillary pressures which vary randomly over square lattices [13], (2) works only for invading fluids of zero viscosity, and (3) disregards interpore surface tension and cannot, therefore, account for the cooperative invasion effects found by Stokes, Kusnick, and Robbins [14] (discussed by Martys, Cieplak, and Robbins [15]) about which nothing further will be said here. We then discuss and determine suitably defined (see below) fractal dimensions, D_i for the IP ($D_i \sim 1.9$), and D_e for the DLA ($D_e = 1.70 \pm 0.04$) regimes. The value of D_i is *not* in disagreement with the value that obtains for it in IP with trapping (see below). Furthermore, we find that the fractal dimension of the interface is given by $D_s = 1.3 \pm 0.1$ for lengths l such that $8 \lesssim l \lesssim L_c/2$, which is close to the value [16] (1.37) found for D_s in IP with trapping. The data for the crossover length L_c are shown to be consistent with Eq. (1). Finally, a scaling argument is given for Eq. (1).

The existence and behavior of a crossover length L_c, from small-length-scale capillary-force controlled growth to large-length-scale viscous fingering in porous media, can be expected on the basis of the following crude argument (for a scaling argument, see further below). Consider some portion of the two-fluid interface on a length scale l, and the pressure difference δp between two points on the viscous fluid which are $\sim l$ apart. Two terms contribute to δp: δp_1, associated with viscous forces, and δp_2, associated with capillary forces. $\delta p_1 \sim |\nabla p| l$, where ∇p is the pressure gradient far downstream. In order to estimate δp_2, keep in mind that pressure variations over the interface on length scales much smaller than l con-

tribute negligibly to δp_2 at distances much larger than l away from it. We therefore expect $\delta p_2 \sim \Delta\tilde{p}(a/l)^{D_s/2}$, where $\Delta\tilde{p}$ is the typical spatial variation of the capillary pressure over a length a (a may be the length of a pore), and D_s is the interface fractal dimension. Setting $\delta p_2 \sim \delta p_1$ gives

$$L_c \sim a|\Delta\tilde{p}/a\nabla p|^{2/(2+D_s)}. \quad (1)$$

We next specify what we mean by a *porous medium* here. A porous medium is often mimicked in the laboratory as a set of cavities (pores) on a lattice connected by channels [8,9] or as a set of randomly packed glass beads [14]. We model it here by a square lattice with random numbers $\tilde{p}_i$, evenly distributed between $-\Delta\tilde{p}/2$ and $\Delta\tilde{p}/2$, assigned to each lattice site i at the outset of the simulation [13]. (Note that a nonvanishing average value of $\tilde{p}$ would only redefine the pressure under which injection takes place, which is a trivial effect.) The pressure drop across the two-fluid interface on site j is given by $\tilde{p}_j$. Spatial variations of the permeability are completely disregarded here.

Our model does not deal with motion within channels or pores. It describes motion on larger scales. Each site in it represents a region of space of about the size of a glass bead (or grain of sand, or whatever). In experiments with *channel networks,* surface tension plays a significant role only *within* channel spaces: Advancing interfaces in different channels are disconnected. It costs therefore little energy to warp interfaces on large scales. That suggests a small value of the surface tension for our model. Capillary pressure, being an intrachannel phenomenon, can be arbitrarily large. On the other hand, there is reasonably good connectivity on the interface if the medium is made up of spherical beads; interpore surface tension may then lead to "cooperative invasion" [15]. We neglect interpore surface tension here, and do not therefore deal with such effects.

We use a DLA model which takes into account capillary pressure rather than solve Laplace's equation numerically because it is a fast algorithm. We can simulate Laplacian growth on lattices of up to 1024×1024 sites. We next describe, following Ref. [7], the DLA algorithm we use here. Let $p' = -p + p_0 + \Delta\tilde{p}/2$ throughout all space, where p_0 is the pressure under which the nonviscous fluid is injected at the bottom of a cell. Clearly, p' satisfies Laplace's equation since (we neglect spatial variations of the permeability) p does. Now, for the boundary conditions, we assume $p=0$ at the top of the cell (where the viscous fluid exists), therefore, $p' = p_0 + \Delta\tilde{p}/2$ there. On the other hand, $p' = \Delta\tilde{p}/2 - p(R)$ at a point R on the interface (on the viscous-fluid side of it). The probabilities P_b and P_t that a random walker be released through a given bond b at R across the interface boundary and from some given site at the top boundary of the cell, respectively, fulfill

$$\frac{P_b(R)}{P_t} = \frac{\Delta\tilde{p}/2 - \tilde{p}(R)}{p_0 + \Delta\tilde{p}/2}.$$

In order to avoid needless walking on large regions of empty space, random walkers are released from a straight line just above the highest point on the interface boundary [7] (instead of from the top of the cell, which is a distance l away) with probability aP_t/l. Letting $l \to \infty$, and $p_0 \to \infty$, keeping $\langle\nabla p\rangle = p_0/l$ constant, gives

$$\frac{P_b(R)}{P_l} = \frac{r}{2}\frac{1-2\tilde{p}(R)}{\Delta\tilde{p}}, \quad (2)$$

where P_l is the probability that the random walker be released from a given point on the straight-line construct and

$$r = \Delta\tilde{p}/a\langle\nabla p\rangle. \quad (3)$$

(Using Darcy's law, $r = \kappa\Delta\tilde{p}/a\tau_0 \mathrm{Ca}$, where κ is the permeability, Ca is the capillary number, and τ_0 is the "bare" surface tension [17]; letting $\Delta\tilde{p} \sim \tau_0/a$ and $\kappa \sim a^2$ gives $r \sim \mathrm{Ca}^{-1}$.) Clearly, r is the only parameter in the problem. Random walks which would break up the aggregate by removing particles from the interface are not allowed.

Results of DLA runs on lattices of $L\times L$ sites for $L=1024$ for $r=0$, 40, 160, and 640 are shown in Figs. 1(a)–1(d), respectively. Blowups of the lower left-hand squares of 256×256 sites of Figs. 1(b)–1(d) are shown in Figs. 1(e)–1(g), respectively. Figure 1(h) is for a 256×256 site lattice and $r=2560$. Figure 1 exhibits two regimes: (1) open branching on large length scales, much as for ordinary DLA [Fig. 1(a)], and (2) IP-like patterns on small length scales (within "branches" and "tree trunks," which clearly become thicker as r increases).

We next measure the fractal dimensions for these two regimes. We use the box-counting method [3]. Only boxes of 4×4 sites or larger are taken into account. In order to keep errors within reasonable bounds, boxes are disregarded if there are fewer than 50 of them [which clearly rules out boxes of $(L/4)\times(L/4)$ sites or larger for lattices of $L\times L$ sites]. As expected, the fractal dimension turns out to be scale dependent. Let D_e (e for exterior) be the fractal dimension for lengths larger than branch thickness L_c, which is defined and determined below. We find $D_e = 1.70 \pm 0.04$ for all $r \leq 320$. We cannot check the value of D_e for $r \geq 640$ because $L_c \gtrsim 100$ then, and, consequently, only boxes of 128×128 sites qualify in lattices of 1024×1024 sites; simulations on larger lattices are necessary to obtain data points for D_e for larger r.

We now turn our attention to the *interior* space of the treelike structures shown in Fig. 1. Unfortunately, the fractal dimension D obtained by counting all nonempty $l\times l$ boxes for $4 < l < L_c/2$, turns out to depend on position and on r. Larger values of D always obtain for a horizontal band, of width of about L_c, at the bottom of the lattice; D seems to be homogeneous otherwise; however, it varies with r, from $D \approx 1.7$ for $r=160$ to $D \approx 1.9$ for $r=2560$. We can obtain a fractal dimension D_i of the interior space, which seems to be approximately independent of position and of r, by counting only boxes which lie

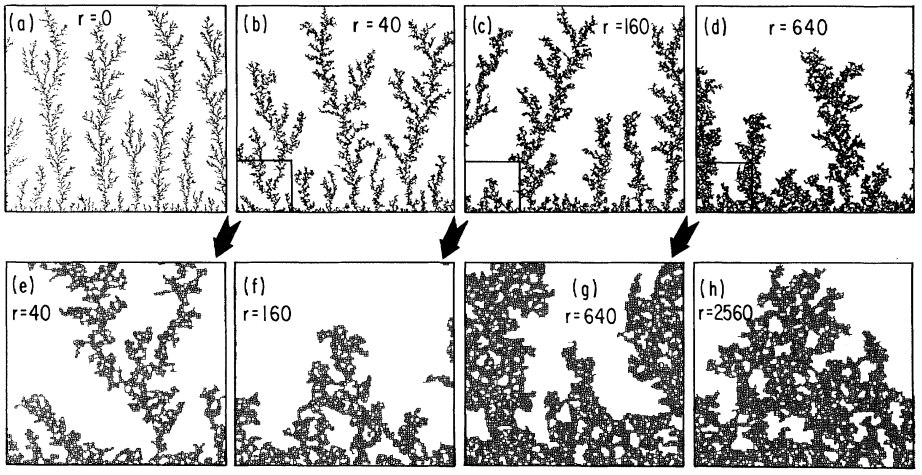

FIG. 1. (a)–(h) Displacement of a viscous fluid by an inviscid one (shown as black) for the values of r [see Eq. (3)] shown. (a)–(d) are for lattices of 1024×1024 sites, (e)–(g) are enlargements of the lower 256×256 left-hand sides of (b)–(d), and (h) is for a run on a 256×256 site lattice for $r=2560$.

completely within interior space (any line joining a point within any such box and the top of the cell must necessarily go through an invaded area). Figure 2(a) shows the results obtained for D_e and for D_i. The data points shown for D_i may be contrasted with $1.80 < D < 1.83$ for IP with trapping [2,8]. However, D and D_i are not the same thing. We have determined D_i for IP (with trapping) data [18] on a 1024×1024 site lattice, and found $D_i - D = 0.07 \pm 0.02$. Furthermore, we find that the fractal dimension of our interfaces is given by $D_s = 1.3 \pm 0.1$ for lengths l such that $8 \lesssim l \lesssim L_c/2$, which is close to the value [16] (1.37) found for D_s in IP with trapping. Consequently, we cannot distinguish our growths, on length scales smaller than L_c, from IP with trapping for $2560 \geq r \geq 160$. (We cannot rule out the possibility that $D_i \to 2$ as $r \to \infty$.)

We now turn our attention to the crossover length L_c. By definition, the branch (and tree trunk) thickness is the length which separates trapping (of the defending fluid) from open branching. Therefore, it provides the natural definition of L_c, which we determine as follows. Take a point i on a growth, draw vertical and horizontal lines through it, and let them end at the exterior space (i.e., all space connected to the top of the cell) boundary. Let the shortest of these two lines be L_i. We let L_c be the average of L_i over the aggregate. Data points obtained are shown in Fig. 2(b). Lattice size effects show up for small values of r, since the branch thickness becomes about $5a$ in the $r \to 0$ limit, instead of vanishing, as it would in the continuum. Meaningful comparisons with the predicted behavior [Eq. (1)] are therefore to be made only for $L_c \gg 5a$. It is gratifying to find that data points in Fig. 2(b) do seem to approach the predicted behavior for $L_c \gtrsim 20a$.

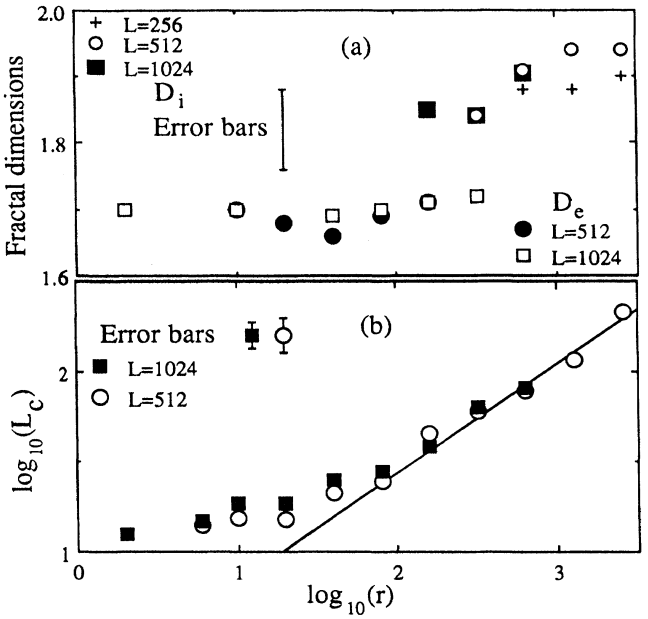

FIG. 2. (a) Plot of large-scale (lengths $> 2L_c$) dimension, D_e, and of interior space dimension, D_i, vs $\log_{10}(r)$. Simulations on larger lattices are needed to extend the set of data points for D_e beyond $r=320$. No data points can be obtained for D_i for $r < 8$ because the length-scale range between "pore size" a and L_c becomes too small then. (b) Plot of $\log_{10}(L_c)$ vs $\log_{10}(r)$ for lattices of $L \times L$ sites for the values of L shown. The straight line is *not* a fit; its slope is 0.606 [$=2/(2+D_s)$, with $D_s=1.3$], as predicted; only its intercept is fitted to the data points. To obtain error bars, each growth was divided into ten horizontal strips and a value of L_c was obtained for each one. The error bars shown stand for twice the value of the corresponding standard deviations.

We next give a scaling argument for L_c. Consider some Laplacian growth and an enlarged version of it with all lengths augmented by α. Speed, and therefore $\kappa\nabla p/\mu$ (through Darcy's law), scales as α. All pressures, including capillary ones, multiplied by κ/μ, must therefore scale as α^2. Now, pores, as everything else, are larger in the blown-up version, but we want to shorten them back to their original size, leaving Laplacian growth unaffected. To do this, $\kappa\Delta\tilde{p}/\mu$ must be augmented by $(n'/n)^{1/2}$, assuming statistical independence of pores, where n'/n is the ratio of small-size to large-size pore numbers on a given portion of the interface. Since $n'/n=\alpha^{D_s}$, by definition of D_s (the fractal dimension by the box method), it follows that $\kappa\Delta\tilde{p}/\mu$ must scale as $\alpha^{2+D_s/2}$. It follows from Eq. (3) that r scales as $\alpha^{1+D_s/2}$. But, we have seen above that L_c depends only on r. Let $L_c=f(r)$; since L_c scales as α (as all other lengths do), it follows that $\alpha f(r)=f(r\alpha^{1+D_s/2})$; consequently, $f(r)=r^{2/(2+D_s)}$. Equation (1), which is the desired relation, follows.

We conclude with two comments. Substitution of $r\sim \mathrm{Ca}^{-1}$ into Eq. (1) gives the simple form, $L_c\sim a\times \mathrm{Ca}^{-2/(2+D_s)}$, whereas the numerical estimate, $L_c\approx 1.67r^{2/(2+D_s)}$, with $D_s=1.3$, follows from a fit of Eq. (1) to the data points in Fig. 2(b). Inspection of Fig. 1 suggests that at least 10 times as many channels as Lenormand [9] had ($\sim 10^5$) are required in order to exhibit experimentally the crossover effect we describe here.

Computations were performed on SUN and IBM workstations. Computing times grow linearly with r. It took a couple of weeks on an IBM Risc/6000mod320 to get Fig. 1(d).

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